Math 270 Day 2

Section 1.2: Solutions and Initial Value Problems

Book clarification

Section 1.2: Solutions and Initial Value Problems

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x^3 \text{ (second-order, } x \text{ independent, } y \text{ dependent)}$$

$$\sqrt{1 - \left(\frac{d^2y}{dt^2}\right)} - y = 0 \text{ (second-order, } t \text{ independent, } y \text{ dependent)}$$

$$\frac{d^4x}{dt^4} = xt \text{ (fourth-order, } t \text{ independent, } x \text{ dependent)}$$

Thus, a general form for an nth-order equation with x independent, y dependent, can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \tag{1}$$

or

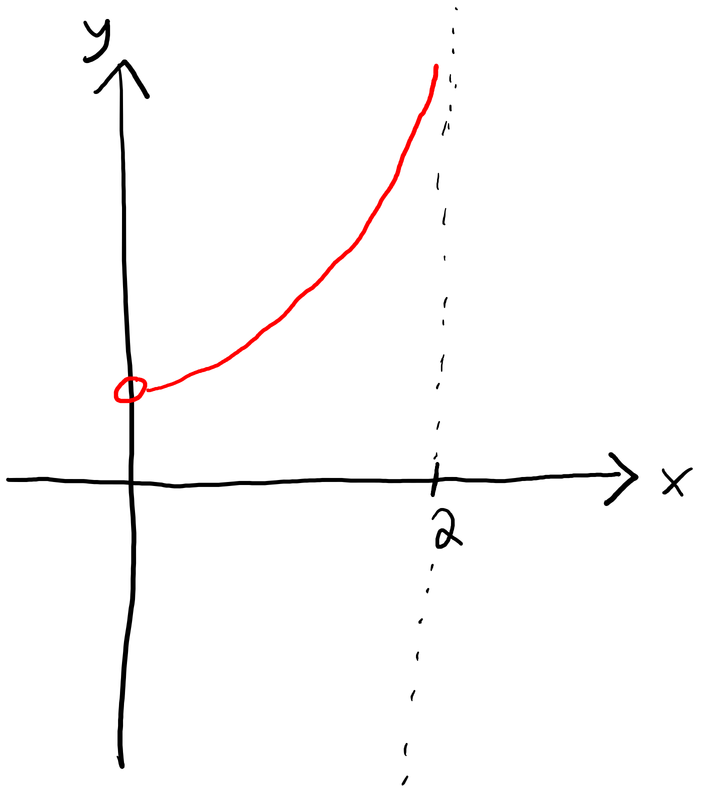
$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right) \tag{2}$$

Section 1.2: Solutions and Initial Value Problems

<u>What is really meant by a solution to a differential equation?</u>

DE: $\frac{dy}{dx} = \frac{y}{\sqrt{x(2-x)}}$

Solution:



- Every point on the curve must satisfy the DE
- Every solution is on an open interval
- A solution isn't always a function, it might be a curve

Section 1.2: Solutions and Initial Value Problems

Checking solutions to DEs

Example 1 Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation $\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0$, but $\psi(x) = x^3$ is not.

Section 1.2: Solutions and Initial Value Problems

Checking solutions to DEs

Example 2 Show that for *any* choice of the constants c_1 and c_2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is an explicit solution to the linear equation $y'' - y' - 2y = 0$.

Section 1.2: Solutions and Initial Value Problems

Checking solutions to DEs

Example 3 Show that the relation $y^2 - x^3 + 8 = 0$ implicitly defines a solution to the nonlinear equation

$$\frac{dy}{dx} = \frac{3x^2}{2y} \quad \text{on the interval } (2, \infty).$$

Section 1.2: Solutions and Initial Value Problems

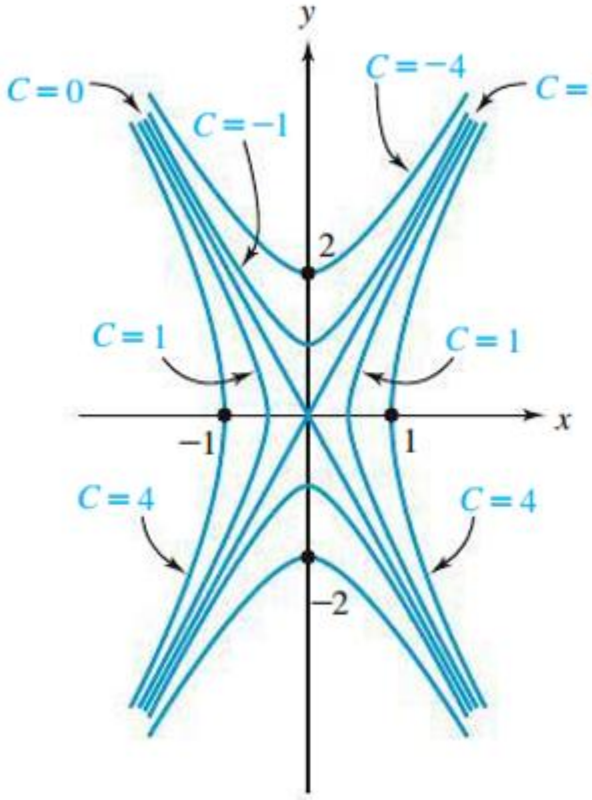
Checking solutions to DEs

Example 4 Show that $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation $(1 + xe^{xy})\dfrac{dy}{dx} + 1 + ye^{xy} = 0$.

Section 1.2: Solutions and Initial Value Problems

Checking solutions to DEs

Example 5 Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to $y\frac{dy}{dx} - 4x = 0$.



Section 1.2: Solutions and Initial Value Problems

Definitions and Theorems

Explicit Solution

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

Implicit Solution

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1) on the interval I if it defines one or more explicit solutions on I.

implicit function theorem

- We will see later that the general solution to an *n*-th order differential equation will have *n* arbitrary constants

Section 1.2: Solutions and Initial Value Problems

Definitions and Theorems

Initial Value Problem

Definition 3. By an **initial value problem** for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0,$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ \frac{dy}{dx}(x_0) &= y_1, \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1}, \end{aligned}$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

Section 1.2: Solutions and Initial Value Problems

Checking solutions to IVPs

Example 6 Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\frac{d^2y}{dx^2} + y = 0\,; \qquad y(0) = -1\,, \qquad \frac{dy}{dx}(0) = 1\,.$$

Section 1.2: Solutions and Initial Value Problems

Checking solutions to IVPs

Example 7 As shown in Example 2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is a solution to $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the initial conditions

$y(0) = 2$ and $\frac{dy}{dx}(0) = -3$ are satisfied.

Section 1.2: Solutions and Initial Value Problems

Definitions and Theorems

Existence and Uniqueness of Solution

Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f / \partial y$ are continuous functions in some rectangle

$$R = \{(x, y): a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.†

Section 1.2: Solutions and Initial Value Problems

Definitions and Theorems

Example 8 For the initial value problem $3\dfrac{dy}{dx} = x^2 - xy^3 \ , \qquad y(1) = 6 \ ,$

does Theorem 1 imply the existence of a unique solution?

Section 1.2: Solutions and Initial Value Problems

Definitions and Theorems

Example 9 For the initial value problem $$\frac{dy}{dx} = 3y^{2/3}\ , \qquad y(2) = 0\ ,$$

does Theorem 1 imply the existence of a unique solution?